

Volkov wave function: its orthonormality and completeness

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The present note aims to provide a clear and explicit derivation of the orthonormality condition, and the completeness property of the Volkov wave function. Thus, none of the results are new.

I. VOLKOV WAVE FUNCTION

The Dirac equation for a charged particle interacting with an arbitrary electromagnetic field is written as

$$i \frac{d|\psi(t)\rangle}{dt} = [\boldsymbol{\alpha} \cdot (\mathbf{p} - e\mathbf{A}) + e\phi + m\beta] |\psi(t)\rangle, \quad (1)$$

where we used the natural units such that $\hbar = c = 1$, e is the charge of the particle, $\boldsymbol{\alpha}$ and β are the Dirac matrices chosen in the standard representation [1], and the gauge potential is given by $A^\mu = (\phi, \mathbf{A})$ with the metric convention $(+, -, -, -)$. For the position representation of the state vector the Dirac equation (1) reads

$$[i\gamma^\mu (\partial_\mu + ieA_\mu) - m] \langle \mathbf{x} | \psi(t) \rangle = 0, \quad (2)$$

with $\gamma^\mu = (\beta, \boldsymbol{\beta}\boldsymbol{\alpha})$.

If we multiple Eq. (2) with the operator $i\gamma^\nu D_\nu + m$ from the left side, we provide

$$0 = [i\gamma^\nu D_\nu + m] [i\gamma^\mu D_\mu - m] \psi(\mathbf{x}, t) = \left[D^2 + \frac{\gamma^\nu \gamma^\mu}{2} [D_\nu, D_\mu] + m^2 \right] \psi(\mathbf{x}, t), \quad (3)$$

with $D_\mu = \partial_\mu + ieA_\mu$ being the covariant derivative, $\psi(\mathbf{x}, t) \equiv \langle \mathbf{x} | \psi(t) \rangle$, and we used the commutation relation $\{\gamma^\nu, \gamma^\mu\} = 2g^{\nu\mu}$. Using the fact that $[D_\nu, D_\mu] = ieF_{\nu\mu}$, where $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the field strength tensor, Eq. (3) can be further written as

$$\left[\partial^2 + 2ieA^\mu \partial_\mu - e^2 A^2 + \frac{ie}{2} \gamma^\nu \gamma^\mu F_{\nu\mu} + m^2 \right] \psi(\mathbf{x}, t) = 0, \quad (4)$$

where we have used the Lorenz gauge condition $\partial_\mu A^\mu = 0$. We observe from Eq. (4) that as the gamma matrices couple to the field strength tensor, the spinor part of the wave function is gauge invariant for an arbitrary electromagnetic field.

Our point of interest is the solution of Eq. (4) for a plane wave. A plane wave can be defined in terms of the following gauge potential $A^\mu = A^\mu(\eta)$ with the phase $\eta = k^\mu x_\mu$, where k^μ is the wave

vector fulfilling the condition $k^2 = 0$. For a plane wave the spin resolved term of Eq. (4) can be given by

$$\frac{ie}{2}\gamma^\nu\gamma^\mu F_{\nu\mu} = ie\gamma^\nu\gamma^\mu\partial_\nu A_\mu = ie\gamma^\nu\gamma^\mu\dot{A}_\mu k_\nu, \quad (5)$$

where the dot denotes derivative with respect to the gauge field's argument η . Now let us seek the solution of Eq. (4) in the form of

$$\psi^{(\pm)}(\mathbf{x}, t) = \exp\left(-i(\pm p^\mu)x_\mu\right) f^{(\pm)}(\eta) u_{p,s}^{(\pm)}, \quad (6)$$

where (+) and (−) label the particle and anti-particle solutions, respectively. The free particle/anti-particle spinor with the normalization $u_{p,s'}^{(\pm)\dagger} u_{p,s}^{(\pm)} = \delta_{s,s'}$ can be given by

$$u_{p,s}^{(+)} = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \phi_s \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma} \phi_s}{p^0 + m} \end{pmatrix}, \quad u_{p,s}^{(-)} = \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma} \chi_s}{p^0 + m} \\ \chi_s \end{pmatrix}, \quad (7)$$

where $p^\mu = (p^0, \mathbf{p})$ is the free particle/anti-particle's four-momentum with the energy $p^0 = \sqrt{m^2 + \mathbf{p}^2} > 0$, $\phi_1 = \chi_2 = (1 \ 0)^T$, $\phi_2 = \chi_1 = (0 \ 1)^T$ are two component spinors, and $\boldsymbol{\sigma}$ are the Pauli matrices [1]. In the absence of the plane wave, $f^{(\pm)}(\eta)$ reduces to the identity, and $\psi^{(\pm)}(\mathbf{x}, t)$ becomes the field-free solution of the Dirac equation. If we plug the ansatz (6) into Eq. (4), the unknown function $f^{(\pm)}(\eta)$ satisfies the following equation

$$\begin{aligned} & \left[-p^2 f^{(\pm)}(\eta) \mp 2ip^\mu k_\mu \dot{f}^{(\pm)}(\eta) + 2ieA^\mu \left(\mp ip_\mu f^{(\pm)}(\eta) + k_\mu \dot{f}^{(\pm)}(\eta) \right) \right. \\ & \left. - e^2 A^2 f^{(\pm)}(\eta) + ie\gamma^\mu k_\mu \gamma^\nu \dot{A}_\nu f^{(\pm)}(\eta) + m^2 f^{(\pm)}(\eta) \right] u_{p,s}^{(\pm)} = 0. \end{aligned} \quad (8)$$

Before going further, we first observe that the Lorenz gauge condition for a plane wave can also be written as $\partial_\mu A^\mu = \dot{A}^\mu k_\mu = d(A^\mu k_\mu)/d\eta$, which implies $A^\mu k_\mu$ equals to a constant, and in fact, without loss of generality, this constant can be set to zero. Using further the on-shell condition $p^2 = m^2$, Eq. 8 simplifies to

$$\left[\mp 2ip^\mu k_\mu \dot{f}^{(\pm)}(\eta) + \left(\pm 2eA^\mu p_\mu - e^2 A^2 + ie\gamma^\mu k_\mu \gamma^\nu \dot{A}_\nu \right) f^{(\pm)}(\eta) \right] u_{p,s}^{(\pm)} = 0, \quad (9)$$

and whose solution just yields

$$f^{(\pm)}(\eta) = \exp\left(\mp i \frac{e}{pk} \int_{\eta_0}^{\eta} d\eta' \left(\pm Ap - e \frac{A^2}{2} \right)\right) \exp\left(\pm e \frac{kA}{2pk}\right), \quad (10)$$

where $k \equiv \gamma^\mu k_\mu$, and η_0 is the initial phase at which the gauge potential and the field vanish. We further notice that

$$\exp\left(\pm e \frac{kA}{2pk}\right) = 1 \pm e \frac{kA}{2pk}, \quad (11)$$

due to the fact that $k^2 = 0$. As a result, the wave function with the quantum numbers p and s reads

$$\psi_{p,s}^{(\pm)}(\mathbf{x}, t) = \exp \left[-i \left(\pm p^\mu x_\mu \pm \frac{e}{pk} \int_{\eta_0}^{\eta} d\eta' \left(\pm Ap - eA^2/2 \right) \right) \right] \left(1 \pm e \frac{kA}{2pk} \right) u_{p,s}^{(\pm)}, \quad (12)$$

which is called the Volkov wave function [2].

II. ORTHONORMALITY OF VOLKOV WAVE FUNCTION

In this section, following Ritus [3], we explicitly show that the Volkov wave function obeys the usual orthonormality condition

$$\int d^3\mathbf{x} \psi_{p',s'}^{(\pm)\dagger}(\mathbf{x}, t) \psi_{p,s}^{(\pm)}(\mathbf{x}, t) = \delta(\mathbf{p}' - \mathbf{p}) \delta_{s',s}. \quad (13)$$

For a shorthand notation, we first rewrite the Volkov wave function in the following form

$$\psi_{p,s}^{(\pm)}(\mathbf{x}, t) = \exp \left[-i \left(\pm S_p^{(\pm)} \right) \right] \left(1 \pm e \frac{kA}{2pk} \right) u_{p,s}^{(\pm)}, \quad (14)$$

with the phase

$$S_p^{(\pm)} \equiv p^\mu x_\mu + \frac{e}{pk} \int_{\eta_0}^{\eta} d\eta' \left(\pm Ap - eA^2/2 \right). \quad (15)$$

Then, we set the scalar part of the gauge potential zero within the Lorenz gauge, and choose the propagation direction along the z -direction and the vector potential on the $x - y$ plane, i.e.,

$$A^\mu(\eta) = (0, A_x(t-z), A_y(t-z), 0). \quad (16)$$

Thereby, the phase of the Volkov wave function is

$$S_p^{(\pm)} = p^0 t - p_z z - \mathbf{p}_\perp \cdot \mathbf{x} + \int_0^{t-z} d\eta' \frac{\mp 2e\mathbf{p}_\perp \cdot \mathbf{A} + e^2 A^2}{2p_-}, \quad (17)$$

where for the sake of convenience we set $\eta_0 = 0$, $\mathbf{p}_\perp = (p_x, p_y)$, and $p_- = p^\mu k_\mu = p^0 - p_z > 0$. The phase can also be rewritten as

$$S_p^{(\pm)} = p^0 t - p_z z - \mathbf{p}_\perp \cdot \mathbf{x} + \int_0^{t-z} d\eta' \frac{(\mathbf{p}_\perp \mp e\mathbf{A})^2 - p_\perp^2}{2p_-} = p_- \frac{t+z}{2} - \mathbf{p}_\perp \cdot \mathbf{x} + \int_0^{t-z} d\eta' \frac{(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2}{2p_-}. \quad (18)$$

As the integrand is independent from x and y , one can immediately evaluate the corresponding integrals in Eq. 13, which are in fact Dirac delta functions. Therefore, the orthonormality reads

$$\begin{aligned} \int d^3\mathbf{x} \psi_{p',s'}^{(\pm)\dagger}(\mathbf{x}, t) \psi_{p,s}^{(\pm)}(\mathbf{x}, t) &= \int d^3\mathbf{x} \exp \left[\pm i (S_{p'}^{(\pm)} - S_p^{(\pm)}) \right] U_{p',s'}^{(\pm)\dagger} U_{p,s}^{(\pm)} \\ &= \delta(\mathbf{p}'_\perp - \mathbf{p}_\perp) \int dz \exp \left[\pm i \left((p'_- - p_-) \frac{t+z}{2} - (p'_- - p_-) \int_0^{t-z} d\eta' \frac{(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2}{2p_- p'_-} \right) \right] U_{p',s'}^{(\pm)\dagger} U_{p,s}^{(\pm)}, \end{aligned} \quad (19)$$

with

$$U_{p,s}^{(\pm)} \equiv \left(1 \pm e \frac{\not{k} \not{A}}{2p_-}\right) u_{p,s}^{(\pm)}. \quad (20)$$

Now, we define

$$\zeta^{(\pm)} = z - \int_0^{t-z} d\eta' \frac{(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2}{p_- p'_-}, \quad (21)$$

which implies

$$d\zeta^{(\pm)} = dz \left(1 + \frac{(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2}{p_- p'_-}\right), \quad (22)$$

and hence simplifies the integral to

$$\begin{aligned} & \int d^3\mathbf{x} \psi_{p',s'}^{(\pm)\dagger}(\mathbf{x},t) \psi_{p,s}^{(\pm)}(\mathbf{x},t) = \delta(\mathbf{p}'_\perp - \mathbf{p}_\perp) \\ & \times \int_{-\infty}^{\infty} d\zeta^{(\pm)} \frac{U_{p',s'}^{(\pm)\dagger} U_{p,s}^{(\pm)}}{1 + [(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2]/p_- p'_-} \exp\left[\pm i(p'_- - p_-) \frac{t + \zeta}{2}\right]. \end{aligned} \quad (23)$$

The spin resolved part can be written as

$$U_{p',s'}^{(\pm)\dagger} U_{p,s}^{(\pm)} = u_{p',s'}^{(\pm)\dagger} \left(1 \pm e \frac{\gamma^0 \not{A} \not{k} \gamma^0}{2p'_-}\right) \left(1 \pm e \frac{\not{k} \not{A}}{2p_-}\right) u_{p,s}^{(\pm)}, \quad (24)$$

and the intermediate square matrix can be decomposed as

$$\left(1 \pm e \frac{\gamma^0 \not{A} \not{k} \gamma^0}{2p'_-}\right) \left(1 \pm e \frac{\not{k} \not{A}}{2p_-}\right) = 1 + \frac{1}{p'_- p_-} \left(e^2 \mathbf{A}^2 \frac{\gamma^0(\gamma^0 - \gamma^3)}{2} \mp 2eA_x \gamma^1 \frac{(p'_- - p_-)\gamma^3 - (p'_- + p_-)\gamma^0}{4} \right. \quad (25)$$

$$\left. \mp 2eA_y \gamma^2 \frac{(p'_- - p_-)\gamma^3 - (p'_- + p_-)\gamma^0}{4} \right). \quad (26)$$

If we evaluate each corresponding term, we provide the following expressions

$$u_{p',s'}^{(\pm)\dagger} u_{p,s}^{(\pm)} = \left(1 + \frac{\mathbf{p}_\perp^2 + m^2}{p'_- p_-}\right) g(p', p) \delta_{s',s}, \quad (27)$$

$$u_{p',s'}^{(\pm)\dagger} \frac{\gamma^0(\gamma^0 - \gamma^3)}{2} u_{p,s}^{(\pm)} = g(p', p) \delta_{s',s}, \quad (28)$$

$$u_{p',s'}^{(\pm)\dagger} \gamma^1 \frac{(p'_- - p_-)\gamma^3 - (p'_- + p_-)\gamma^0}{4} u_{p,s}^{(\pm)} = p_x g(p', p) \delta_{s',s}, \quad (29)$$

$$u_{p',s'}^{(\pm)\dagger} \gamma^2 \frac{(p'_- - p_-)\gamma^3 - (p'_- + p_-)\gamma^0}{4} u_{p,s}^{(\pm)} = p_y g(p', p) \delta_{s',s}, \quad (30)$$

where the common function

$$g(p', p) = \frac{(p'_- + m)(p_- + m) + \mathbf{p}_\perp^2}{4 \sqrt{(m + p^0)(m + p'^0) p^0 p'^0}}. \quad (31)$$

As a result, we obtain

$$U_{p',s'}^{(\pm)\dagger} U_{p,s}^{(\pm)} = \left(1 + [(\mathbf{p}_\perp \mp e\mathbf{A})^2 + m^2] / p_- p'_-\right) g(p', p) \delta_{s',s}, \quad (32)$$

which exactly cancels the corresponding denominator in Eq. (23), and therefore we find

$$\int d^3\mathbf{x} \psi_{p',s'}^{(\pm)\dagger}(\mathbf{x}, t) \psi_{p,s}^{(\pm)}(\mathbf{x}, t) = \delta(\mathbf{p}'_\perp - \mathbf{p}_\perp) \delta\left(\frac{p'_- - p_-}{2}\right) g(p, p') \delta_{s',s}. \quad (33)$$

The function $g(p, p')$ reduces to

$$g(p, p') = \frac{p_-}{2p^0} \quad (34)$$

for the same momentum. Finally, using the relation

$$\frac{p_-}{2p^0} \delta\left(\frac{p'_- - p_-}{2}\right) = \delta(p'_z - p_z), \quad (35)$$

we show that

$$\int d^3\mathbf{x} \psi_{p',s'}^{(\pm)\dagger}(\mathbf{x}, t) \psi_{p,s}^{(\pm)}(\mathbf{x}, t) = \delta(\mathbf{p}' - \mathbf{p}) \delta_{s',s}. \quad (36)$$

III. COMPLETENESS OF VOLKOV WAVE FUNCTION

In this section, we present a clear derivation of the completeness property of the Volkov wave function, explicitly we will show that

$$C(\mathbf{x}, \mathbf{x}') = \sum_{s=1}^2 \int d^3\mathbf{p} \left(\psi_{p,s}^{(+)}(\mathbf{x}, t) \psi_{p,s}^{(+)\dagger}(\mathbf{x}', t) + \psi_{p,s}^{(-)}(\mathbf{x}, t) \psi_{p,s}^{(-)\dagger}(\mathbf{x}', t) \right) = \delta(\mathbf{x} - \mathbf{x}'). \quad (37)$$

Using the form of the Volkov wave function given in Eq. (14), the completeness can be given by

$$\begin{aligned} C(\mathbf{x}, \mathbf{x}') &= \int d^3\mathbf{p} \exp[-iS^{(+)}(\mathbf{x}) + iS^{(+)}(\mathbf{x}')] U_{p,s}^{(+)}(\mathbf{x}) U_{p,s}^{(+)\dagger}(\mathbf{x}') \\ &\quad + \int d^3\mathbf{p} \exp[iS^{(-)}(\mathbf{x}) - iS^{(-)}(\mathbf{x}')] U_{p,s}^{(-)}(\mathbf{x}) U_{p,s}^{(-)\dagger}(\mathbf{x}'), \end{aligned} \quad (38)$$

with

$$U_{p,s}^{(\pm)}(\mathbf{x}) U_{p,s}^{(\pm)\dagger}(\mathbf{x}') = \left(1 \pm e^{\frac{\mathbf{k}\mathbf{A}(\mathbf{x})}{2p_-}}\right) \sum_s u_{p,s}^{(\pm)} u_{p,s}^{(\pm)\dagger} \left(1 \pm e^{\frac{\gamma^0 \mathbf{A}(\mathbf{x}') \mathbf{k} \gamma^0}{2p'_-}}\right) \quad (39)$$

$$= \left(1 \pm e^{\frac{\mathbf{k}\mathbf{A}(\mathbf{x})}{2p_-}}\right) \frac{\not{p} \pm m}{2p^0} \gamma^0 \left(1 \pm e^{\frac{\gamma^0 \mathbf{A}(\mathbf{x}') \mathbf{k} \gamma^0}{2p'_-}}\right), \quad (40)$$

where we used the completeness of the free particle/anti-particle spinor in the second line. With the help of the phase (18), Eq. (38) is rewritten as

$$C(\mathbf{x}, \mathbf{x}') = \int d^3 \mathbf{p} e^{-i \left[\frac{p_-}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_-} \right]} U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') \\ + \int d^3 \mathbf{p} e^{i \left[\frac{p_-}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp + e\mathbf{A})^2 + m^2}{2p_-} \right]} U_{\mathbf{p},s}^{(-)}(\mathbf{x}) U_{\mathbf{p},s}^{(-)\dagger}(\mathbf{x}'). \quad (41)$$

After the following transformation $\mathbf{p} \rightarrow -\mathbf{p}$ in the second integral, we provide

$$C(\mathbf{x}, \mathbf{x}') = \int d^3 \mathbf{p} e^{-i \left[\frac{p_-}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_-} \right]} U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') \\ + \int d^3 \mathbf{p} e^{-i \left[-\frac{p_+}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') + \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_+} \right]} U_{-\mathbf{p},s}^{(-)}(\mathbf{x}) U_{-\mathbf{p},s}^{(-)\dagger}(\mathbf{x}'), \quad (42)$$

with $p_+ = p^0 + p_z$. Now, for the first integral we apply the coordinate transformation $(\mathbf{p}) \rightarrow (\mathbf{p}_\perp, p_-)$, and for the second integral $(\mathbf{p}) \rightarrow (\mathbf{p}_\perp, p_+)$. In other words, for the first integral

$$p_z = \frac{m^2 + \mathbf{p}_\perp^2 - p_-^2}{2p_-} \rightarrow \int_{-\infty}^{\infty} dp_z = \int_0^{\infty} dp_- \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_-^2}{2p_-^2} \right), \quad (43)$$

and for the second integral

$$p_z = \frac{m^2 + \mathbf{p}_\perp^2 - p_+^2}{-2p_+} \rightarrow \int_{-\infty}^{\infty} dp_z = \int_0^{\infty} dp_+ \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_+^2}{2p_+^2} \right). \quad (44)$$

Then, the completeness property becomes

$$C(\mathbf{x}, \mathbf{x}') = \int d^2 \mathbf{p}_\perp \int_0^{\infty} dp_- \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_-^2}{2p_-^2} \right) e^{-i \left[\frac{p_-}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_-} \right]} \\ \times U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') \\ + \int d^2 \mathbf{p}_\perp \int_0^{\infty} dp_+ \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_+^2}{2p_+^2} \right) e^{-i \left[-\frac{p_+}{2}(z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') + \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_+} \right]} \\ \times U_{-\mathbf{p},s}^{(-)}(\mathbf{x}) U_{-\mathbf{p},s}^{(-)\dagger}(\mathbf{x}'). \quad (45)$$

If we further apply the transformation $p_+ \rightarrow -p_-$ in the second integral, we provide

$$\begin{aligned}
C(\mathbf{x}, \mathbf{x}') = & \int d^2 \mathbf{p}_\perp \int_0^\infty dp_- \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_-^2}{2p_-^2} \right) e^{-i \left[\frac{p_-}{2} (z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_-} \right]} \\
& \times U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') \\
& + \int d^2 \mathbf{p}_\perp \int_{-\infty}^0 dp_- \left(\frac{m^2 + \mathbf{p}_\perp^2 + p_-^2}{2p_-^2} \right) e^{-i \left[\frac{p_-}{2} (z - z') - \mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}') - \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p_-} \right]} \\
& \times U_{-\mathbf{p},s}^{(-)}(\mathbf{x}) U_{-\mathbf{p},s}^{(-)\dagger}(\mathbf{x}') \Big|_{p_+ = -p_-}.
\end{aligned} \tag{46}$$

At this level we observe that

$$U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') = U_{-\mathbf{p},s}^{(-)}(\mathbf{x}) U_{-\mathbf{p},s}^{(-)\dagger}(\mathbf{x}') \Big|_{p_+ = -p_-}, \tag{47}$$

therefore, the completeness just yields

$$C(\mathbf{x}, \mathbf{x}') = \int d^2 \mathbf{p}_\perp \exp(i\mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}')) \int_{-\infty}^\infty dp_- \exp \left[-i \left(\kappa_1 p_- - \frac{\kappa_2}{p_-} \right) \right] \frac{p_-^0}{p_-} U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}'), \tag{48}$$

where we used $p^0 = (m^2 + \mathbf{p}_\perp^2 + p_-^2)/(2p_-)$, and

$$\kappa_1 \equiv \frac{z - z'}{2}, \quad \kappa_2 \equiv \int_{t-z}^{t-z'} \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2} \tag{49}$$

such that $\kappa_1 \kappa_2 \geq 0$.

The spin resolved part (47), on the other hand, can be written as

$$\begin{aligned}
& \frac{p_-^0}{p_-} U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') = \\
& \frac{1}{2p_-} \left\{ \not{p} + m + \frac{e}{2p_-} \left[(A(z') \not{k} - A(z) \not{k})(\not{p} + m) - 2p_- A(z') + 2p A(z') \not{k} - e \not{k} A(z) A(z') \right] \right\} \gamma^0.
\end{aligned} \tag{50}$$

We notice that the above equation is in the form of

$$\frac{p_-^0}{p_-} U_{\mathbf{p},s}^{(+)}(\mathbf{x}) U_{\mathbf{p},s}^{(+)\dagger}(\mathbf{x}') = K_0 + \frac{K_1}{p_-} + \frac{K_2}{p_-^2}, \tag{51}$$

where

$$K_0 = \frac{\gamma^0 + \gamma^3}{4} \gamma^0, \tag{52a}$$

$$K_1 = \frac{1}{2} \left[m - \mathbf{p}_\perp \cdot \boldsymbol{\gamma} + \frac{e}{2} [A(z') - A(z)] \gamma^0 (\gamma^0 + \gamma^3) - e A(z') \right] \gamma^0, \tag{52b}$$

$$K_2 = \frac{1}{4} \left[m^2 + \mathbf{p}_\perp^2 + e [A(z') - A(z)] \mathbf{p}_\perp \cdot \boldsymbol{\gamma} + e m [A(z') - A(z)] - e 2\mathbf{p}_\perp \cdot \mathbf{A}(z) - e^2 A(z) A(z') \right] \not{k} \gamma^0. \tag{52c}$$

Finally, using the expression (see Appendix IV)

$$\int_{-\infty}^{\infty} dp_- \frac{\exp\left[-i\left(\kappa_1 p_- - \frac{\kappa_2}{p_-}\right)\right]}{p_-^n} = \begin{cases} \delta(\kappa_1) & n = 0 \\ 0 & n = 1 \\ \delta(\kappa_2) & n = 2 \end{cases}, \quad (53)$$

with

$$\delta(\kappa_1) = 2\delta(z - z'), \quad \delta(\kappa_2) = \frac{2\delta(z - z')}{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}, \quad (54)$$

we prove that the completeness property can be written as

$$\begin{aligned} C(\mathbf{x}, \mathbf{x}') &= \int d^2 \mathbf{p}_\perp \exp(i\mathbf{p}_\perp \cdot (\mathbf{x} - \mathbf{x}')) \left[\frac{\delta(\kappa_2)}{4} ((\mathbf{p}_\perp - e\mathbf{A})^2 + m^2) (\gamma^0 - \gamma^3) + \frac{\delta(\kappa_1)}{4} (\gamma^0 + \gamma^3) \right] \gamma^0 \\ &= \delta(\mathbf{x} - \mathbf{x}'). \end{aligned} \quad (55)$$

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IV. APPENDIX

We will evaluate the following integral

$$I_n \equiv \int_{-\infty}^{\infty} dx f_n(x) \equiv \int_{-\infty}^{\infty} dx \frac{\exp\left[-i\left(\kappa_1 x - \frac{\kappa_2}{x}\right)\right]}{x^n}, \quad (56)$$

with $\kappa_1 \geq 0$, and $\kappa_2 \geq 0$. Accordingly, we consider the corresponding complex integral along the contour shown in Fig.1

$$\oint_C dz f_n(z) = \oint_C dz \frac{\exp\left[-i\left(\kappa_1 z - \frac{\kappa_2}{z}\right)\right]}{z^n} = 0, \quad (57)$$

where the second equality is due to that there is no pole inside the contour. The complex integral can be decomposed as

$$\oint_C dz f_n(z) = I_n + \lim_{\rho \rightarrow 0^+} \int_{C_\rho} dz f_n(z) + \lim_{R \rightarrow \infty} \int_{C_R} dz f_n(z) = 0, \quad (58)$$

then the integral I_n can be given by

$$I_n = - \lim_{\rho \rightarrow 0^+} \left[\int_0^\pi \frac{id\theta}{\rho^{n-1} e^{i(n-1)(\theta+\pi)}} \exp \left(i\kappa_1 \rho \cos(\theta) - \kappa_1 \rho \sin(\theta) - \frac{i\kappa_2}{\rho} \cos(\theta) - \frac{\kappa_2}{\rho} \sin(\theta) \right) \right] \\ + \lim_{R \rightarrow \infty} \left[\int_0^\pi \frac{id\theta}{R^{n-1} e^{-i(n-1)\theta}} \exp \left(-i\kappa_1 R \cos(\theta) - \kappa_1 R \sin(\theta) + \frac{i\kappa_2}{R} \cos(\theta) - \frac{\kappa_2}{R} \sin(\theta) \right) \right]. \quad (59)$$

We first observe that when $n = 1$, the integral vanishes, $I_1 = 0$. In the case of $n = 0$, the integral along the contour C_ρ vanishes, and the integral I_0 becomes

$$I_0 = \lim_{R \rightarrow \infty} \left[\int_0^\pi id\theta R e^{-i\theta} \exp(-i\kappa_1 R \cos(\theta) - \kappa_1 R \sin(\theta)) \right], \quad (60)$$

which vanishes as long as $\kappa_1 \neq 0$. In the case of $\kappa_1 = 0$, the integral yields $I_0 = \lim_{R \rightarrow \infty} 2R$, which goes to infinity, as a result $I_0 = \delta(\kappa_1)$. In a similar way, when $n = 2$, the integral along the contour C_R goes to zero, and we have

$$I_2 = \lim_{\rho \rightarrow 0^+} \left[\int_0^\pi \frac{id\theta}{\rho} e^{-i\theta} \exp \left(-\frac{i\kappa_2}{\rho} \cos(\theta) - \frac{\kappa_2}{\rho} \sin(\theta) \right) \right], \quad (61)$$

which is zero for $\kappa_2 \neq 0$, otherwise $I_2 = \lim_{\rho \rightarrow 0^+} 2/\rho$, i.e., $I_2 = \delta(\kappa_2)$. To sum up, we conclude

$$\int_{-\infty}^{\infty} dx \frac{\exp \left[-i \left(\kappa_1 x - \frac{\kappa_2}{x} \right) \right]}{x^n} = \begin{cases} \delta(\kappa_1) & n = 0 \\ 0 & n = 1 \\ \delta(\kappa_2) & n = 2 \end{cases}. \quad (62)$$

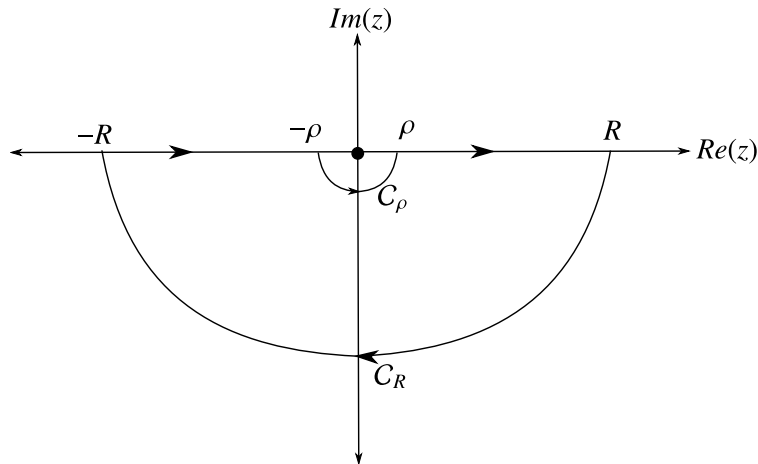


FIG. 1. The contour for $\kappa_1 \geq 0$, and $\kappa_2 \geq 0$ is shown. In the opposite case, the contour can be chosen in the upper-half plane.

We note that in the case of $\kappa_1 \leq 0$, and $\kappa_2 \leq 0$, which also satisfies $\kappa_1 \kappa_2 \geq 0$, one can chose the integration contour in the upper-half plane of the complex plane, see for a detailed discussion Ref. [4].

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